

Characters of Finite Quasigroups IV: Products and Superschemes

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For finite loops (as for finite groups), the character table of a direct product is the tensor product of the character tables of the direct factors. This is no longer true for quasigroups. Although non-3 and 3-quasigroups may have the same character table, the character table of $Q \times Q$ determines whether a finite non-empty quasigroup Q lies in 3 or not. A combinatorial interpretation of the tensor square of a quasigroup character table is obtained in terms of superschemes, a higher-dimensional extension of the concept of association scheme.

1. INTRODUCTION

This paper is part of a series ([6]–[9]) developing the combinatorial character theory of finite quasigroups and the theory of association schemes on which it depends. The current paper is concerned with character tables of direct products and tensor products of character tables. For groups, these two concerns coincide: the character table of a direct product of groups is the tensor product of the character tables of the direct factors. Theorem 2.1 shows that this result is still true for loops, although one cannot use the group-theoretical proof which observes that representations of the direct product are tensor products of representations of the factors. Corollary 2.2 notes that for a finite loop Q with character table Ψ , the character table of the direct square $Q \times Q$ is the tensor square $\Psi \otimes \Psi$. For general finite non-empty quasigroups Q , even this restricted result fails. There are pairs of quasigroups Q_1, Q_2 having the same character table for which the direct squares $Q_1 \times Q_1$ and $Q_2 \times Q_2$ have different character tables. This phenomenon is examined in section 3. It is connected with the class 3 of quasigroups Q for which the diagonal \hat{Q} is a normal subquasigroup of the direct square $Q \times Q$. Theorem 3.1 shows that such quasigroups Q may be recognised by the character table of $Q \times Q$, although [9, Proposition 7.2(b)] showed that they cannot be recognised by the character table of Q .

Since the tensor square $\Psi \otimes \Psi$ of the character table Ψ of a finite non-empty quasigroup Q is not the character table of $Q \times Q$ in general, a different combinatorial interpretation for $\Psi \otimes \Psi$ has to be found. This is done in Theorem 7.1. The interpretation involves the concept of superscheme. A superscheme on a finite set Q is a particular set of relations of arbitrary arity, comprising an association scheme as its subset of binary relations. This association scheme is called the ‘associated scheme’ of the superscheme. The basic general definitions are given in section 4. Just as an association scheme determines a commutative algebra, its Bose–Mesner algebra, so a superscheme determines a graded algebra called the ‘superalgebra’ of the superscheme. This superalgebra, and each of its homogeneous components, are two-sided modules over the Bose–Mesner algebra of the associated scheme. Superalgebras are examined in general in sections 5 and 6. Theorem 6.2 shows that the tensor square of the Bose–Mesner algebra is described faithfully by its action on a homogeneous component of positive degree in the superalgebra. For the finite non-empty quasigroup Q with multiplication group G , the orbits of G on the various direct powers of Q determine a superscheme. Applying Theorem 6.2 to this superscheme then gives the combinatorial interpretation (Theorem 7.1) of the tensor square $\Psi \otimes \Psi$ of the character table Ψ of Q .

2. DIRECT PRODUCTS OF LOOPS

If P and Q are groups with character tables Φ and Ψ respectively, then the direct product $P \times Q$ has character table $\Phi \otimes \Psi$ [10, §3.2]. The usual proof is that each irreducible representation of $P \times Q$ is the tensor product of an irreducible representation of P with an irreducible representation of Q . If P and Q are loops, then the main Theorem 2.1 of this section shows that the character table of $P \times Q$ is still $\Phi \otimes \Psi$. However, the group theory proof using representations is no longer available.

THEOREM 2.1. *Let P and Q be loops with respective character tables Φ and Ψ . Then the loop $P \times Q$ has character table $\Phi \otimes \Psi$.*

PROOF. Let P have multiplication group H and identity element f . Let Q have multiplication group G and identity element e . For (p, q) in $P \times Q$, one has $L((p, q)) = L((p, e))L((f, q)) = L((f, q))L((p, e))$ and $R((p, q)) = R((p, e))R((f, q)) = R((f, q))R((p, e))$. Furthermore, $[L((p, e)), R((f, q))] = [R((p, e)), L((f, q))] = 1$. Thus the multiplication group of $P \times Q$ is $H \times G$. Let the orbits of H in its diagonal action \hat{H} on $P \times P$ be $D_1 = \hat{P}, \dots, D_t$, with incidence matrices $\mathbf{D}_1, \dots, \mathbf{D}_t$ (cf. [6, §2]). Let the orbits of G in its diagonal action \hat{G} on $Q \times Q$ be $C_1 = \hat{Q}, \dots, C_s$, with incidence matrices $\mathbf{C}_1, \dots, \mathbf{C}_s$. Then the orbits of $H \times G$ in its diagonal action on $P^2 \times Q^2$ are of the form $D_j \times C_i$, with $1 \leq j \leq t, 1 \leq i \leq s$. The incidence matrices of these orbits are the Kronecker products $\mathbf{D}_j \otimes \mathbf{C}_i$.

The incidence matrices $\mathbf{D}_1, \dots, \mathbf{D}_t$ span the centralizer ring $V(H, P)$, a commutative subalgebra of the $|P| \times |P|$ complex matrix algebra. The centralizer ring has a new basis $\{\mathbf{F}_1, \dots, \mathbf{F}_t\}$ of orthogonal idempotents, related to the original basis $\{\mathbf{D}_1, \dots, \mathbf{D}_t\}$ by coefficients v_{jm} , with $\mathbf{D}_j = \sum_{m=1}^t v_{jm} \mathbf{F}_m$. Further, Φ is the $t \times t$ matrix of weighted coefficients $\phi_{jm} = |P|(\text{tr } \mathbf{F}_j)^{1/2} v_{mj} |D_m|^{-1}$. Similarly, $V(G, Q)$ has bases $\{\mathbf{C}_1, \dots, \mathbf{C}_s\}$ and $\{\mathbf{E}_1, \dots, \mathbf{E}_s\}$ with $\mathbf{c}_i = \sum_{l=1}^s \xi_{il} \mathbf{E}_l$, where $\mathbf{E}_1, \dots, \mathbf{E}_s$ are orthogonal idempotents and Ψ is the $s \times s$ matrix of weighted coefficients $\psi_{il} = |Q|(\text{tr } \mathbf{E}_i)^{1/2} \xi_{il} |C_l|^{-1}$. The span of the set $\{\mathbf{D}_j \otimes \mathbf{C}_i | 1 \leq j \leq t, 1 \leq i \leq s\}$ of incidence matrices is the subalgebra $V(H \times G, P \times Q) = V(H, P) \otimes V(G, Q)$ of the $|P| \cdot |Q| \times |P| \cdot |Q|$ complex matrix algebra. The set $\{\mathbf{F}_m \otimes \mathbf{E}_l | 1 \leq m \leq t, 1 \leq l \leq s\}$ of orthogonal idempotents is a second basis for $V(H \times G, P \times Q)$, with

$$\mathbf{D}_j \otimes \mathbf{C}_i = \sum_{m=1}^t \sum_{l=1}^s v_{jm} \xi_{il} \mathbf{F}_m \otimes \mathbf{E}_l. \quad (2.1)$$

Let $\Omega = (\omega_{ji,ml} | 1 \leq j, m \leq t, 1 \leq i, l \leq s)$ be the character table of $P \times Q$. Then, by (2.1),

$$\begin{aligned} \omega_{ji,ml} &= |P \times Q|(\text{tr } \mathbf{F}_j \otimes \mathbf{E}_i)^{1/2} v_{mj} \xi_{il} |D_m \times C_l|^{-1} \\ &= |P|(\text{tr } \mathbf{F}_j)^{1/2} v_{mj} |D_m|^{-1} \cdot |Q|(\text{tr } \mathbf{E}_i)^{1/2} \xi_{il} |C_l|^{-1} \\ &= \phi_{jm} \psi_{il}. \end{aligned}$$

Thus $\Omega = \Phi \otimes \Psi$, as required.

COROLLARY 2.2. *If Q is a loop with character table Ψ , then the character table of $Q \times Q$ is $\Psi \otimes \Psi$.*

3. 3-QUASIGROUPS

If P and Q are quasigroups with character tables Φ and Ψ respectively, then it need not be true that the character table of $P \times Q$ is $\Phi \otimes \Psi$. Indeed, even Corollary 2.2 fails

for quasigroups in a very strong sense: the character table of a quasigroup Q need not determine the character table of $Q \times Q$. This failure is connected with the problem of recognising 3-quasigroups from character tables. Recall that Q is a 3-quasigroup if the diagonal \bar{Q} is a normal subquasigroup of (i.e. congruence class under a congruence on) the direct square $Q \times Q$. In a certain sense, 3-quasigroups amongst quasigroups play a similar role to that of abelian groups amongst groups. However, although abelian groups may be recognised by their character tables (even amongst quasigroups [9, Prop. 7.2(a)]), it was shown in [9, Prop. 7.2(b)] that 3-quasigroups cannot be recognised by their character tables. There are two rank-2 quasigroups of order 5, each having the character table [9, (7.1)], one of which is in the class 3 and the other of which is not. But 3-quasigroups can be recognised by the character tables of their direct squares.

THEOREM 3.1. *The character table of the direct square $Q \times Q$ of a finite non-empty quasigroup Q determines whether or not Q is a 3-quasigroup.*

PROOF. By [4], [5], a quasigroup Q is a 3-quasigroup iff the congruence lattice of $Q \times Q$ contains a common complement to the kernels of the two projections $\pi_i: Q \times Q \rightarrow Q$; $(q_1, q_2) \mapsto q_i, i = 1, 2$. But by [6, Theorem 3.6] [11, p. 545], the character table of $Q \times Q$ specifies the congruence lattice of $Q \times Q$.

COROLLARY 3.2. *The character table of a quasigroup Q does not specify the character table of the direct square $Q \times Q$ in general.*

Let Q be a quasigroup with character table Ψ . If Q is a loop, then the tensor square $\Psi \otimes \Psi$ is the character table of $Q \times Q$ by Corollary 2.2. If Q is not a loop, Corollary 3.2 shows that this interpretation of $\Psi \otimes \Psi$ is not available. For a generally valid combinatorial interpretation of the tensor square (Theorem 7.1), one must look to a higher-dimensional extension of the basic concepts of association scheme and Bose–Mesner algebra.

4. SUPERSCHMES

This section and the two subsequent ones propose an extension to arbitrary dimensions of the concepts of association scheme and Bose–Mesner algebra. The setting is quite general, but restricted to the context of finite non-empty quasigroups. To begin with, recall the basic definition of an association scheme (as in [2, II.2.2], [3, §2.1], [7, §3]).

DEFINITION 4.1. Let Q be a finite non-empty set. Then an *association scheme* (Q, Γ) on Q is a partition $\Gamma = \{C_1, \dots, C_s\}$ of the direct square Q^2 such that:

- (A1) $C_1 = \{(x, x) \in Q^2 \mid x \in Q\}$;
- (A2) the converse of each relation in Γ belongs to Γ ;
- (A3) $\forall C_i \in \Gamma, \forall C_j \in \Gamma, \forall C_k \in \Gamma, \exists c_{ijk} \in N. \forall (x, y) \in C_k,$
 $|\{z \in Q \mid (x, z) \in C_i, (z, y) \in C_j\}| = c_{ijk};$
- (A4) $\forall 1 \leq i, j, k \leq s, c_{ijk} = c_{jik}.$

The definition of a superscheme, the higher-dimensional extension of the notion of association scheme, is then given as follows.

DEFINITION 4.2. Let Q be a finite non-empty set. Then a *superscheme* (Q, Γ^*) on Q is a partition $\Gamma^n = \{C_1^n, \dots, C_{s_n}^n\}$ of the direct power Q^{n+2} , for each natural number

n , such that:

$$(S1) \quad C_1^0 = \{(x, x) \in Q^2 \mid x \in Q\};$$

$$(S2) \quad \text{for each function } f: \{1, \dots, m+2\} \rightarrow \{1, \dots, n+2\},$$

and for each relation C_j^n in Γ^n , the relation

$$f^*(C_j^n) = \{(x_1, \dots, x_{m+2}) \mid \exists (y_1, \dots, y_{n+2}) \in C_j^n. \forall 1 \leq i \leq m+2, x_i = y_{if}\}$$

is an element of Γ^m ;

$$(S3) \quad \forall m \geq 0, \forall n \geq 0, \forall C_i^m \in \Gamma^m, \forall C_j^n \in \Gamma^n, \forall C_k^{m+n} \in \Gamma^{m+n}, \exists c(i, j, k; m, n) \in \mathbb{N}.$$

$$\forall (x_0, \dots, x_m, y_0, \dots, y_n) \in C_k^{m+n}, \{z \in Q \mid (x_0, \dots, x_m, z) \in C_i^m,$$

$$(z, y_0, \dots, y_n) \in C_j^n\} = c(i, j, k; m, n);$$

$$(S4) \quad \forall 1 \leq i, j, k \leq s_0, c(i, j, k; 0, 0) = c(j, i, k; 0, 0).$$

REMARKS 4.3. (a) If (Q, Γ^*) is a superscheme, then (Q, Γ^0) is an association scheme called the *associated scheme* of the superscheme. Condition (S2) for the bijection $f: 1 \mapsto 2, 2 \mapsto 1$ reduces to condition (A2).

(b) The function $s: \mathbb{N} \rightarrow \mathbb{Z}^+$; $n \mapsto s_n$ is called the *dimension sequence* of the superscheme (Q, Γ^*) . The exponential generating function $f(x) = \sum_{n=1}^{\infty} s_{n-1} x^n / n!$ is called the *Poincaré series* of (Q, Γ^*) .

(c) The set of superschemes on a given finite non-empty set Q is partially ordered, with $(Q, \Gamma^*) \leq (Q, \Delta^*)$ iff for each $n \geq 0$, $\Gamma^n \leq \Delta^n$ in the partition lattice $\Pi(Q^{n+2})$ [cf. 11, I.2B]. Let $f(x)$ and $g(x)$ respectively be the Poincaré series of the maximal and minimal superschemes on a set Q of cardinality n . Then $f(\ln(1+x)) = (1+x)^{\sum_{k=0}^{n-1} x^k / k!}$ and $g(\ln(1+x)) = (1+x)^n - 1$.

(d) The name 'superscheme' describes the erection of a superstructure on the associated scheme, and reflects the grading n (as in the usage 'Lie superalgebra' for 'graded Lie algebra').

5. SUPERALGEBRAS

Let (Q, Γ^*) be a superscheme on a finite non-empty set Q . For each natural number n , take a complex vector space A^n with basis Γ^n . Let $A = \bigoplus_{n \geq 0} A^n$, the complex vector space direct sum of the A^n . Then A carries an algebra structure defined by

$$C_i^m C_j^n = \sum_{k=1}^{s_{m+n}} c(i, j, k; m, n) C_k^{m+n}. \quad (5.1)$$

Note that A^0 forms a subalgebra isomorphic to the Bose–Mesner algebra of the associated scheme (Q, Γ^0) . The algebra A is called the *Bose–Mesner superalgebra* of the superscheme (Q, Γ) . Since $A^m A^n \subseteq A^{m+n}$ by (5.1), the superalgebra is graded. The subspace A^n is called the *homogeneous component of degree n* .

THEOREM 5.1. *The Bose–Mesner superalgebra of a superscheme is associative.*

PROOF. It suffices to prove

$$(C_i^m C_j^n) C_k^p = C_i^m (C_j^n C_k^p). \quad (5.2)$$

The coefficient of C_l^{m+n+p} in the l.h.s. of (5.2) is

$$\sum_{q=1}^{s_{m+n}} c(i, j, q; m, n) c(q, k, l; m+n, p). \quad (5.3)$$

The coefficient of C_l^{m+n+p} in the r.h.s. of (5.2) is

$$\sum_{r=1}^{s_{n+p}} c(j, k, r; n, p) c(i, r, l; m, n + p). \quad (5.4)$$

Fix an element $(x_0, \dots, x_m, y_1, \dots, y_n, z_0, \dots, z_p)$ of C_l^{m+n+p} . Then, by (S3), both (5.3) and (5.4) count the number of elements (t, u) of Q^2 such that (x_0, \dots, x_m, t) lies in C_i^m , (t, y_1, \dots, y_n, u) lies in C_j^n , and (u, z_0, \dots, z_p) lies in C_k^p .

COROLLARY 5.2. *Each homogeneous component of the Bose–Mesner superalgebra of a superscheme is a two-sided module over the commutative Bose–Mesner algebra of the associated scheme.*

PROOF. The left and right actions of A^0 on a homogeneous component A^m are given by

$$L: A^0 \rightarrow \text{End}_{\mathbb{C}} A^m; x \mapsto (y \mapsto xy) \quad (5.5)$$

and

$$R: A^0 \rightarrow \text{End}_{\mathbb{C}} A^m; x \mapsto (y \mapsto yx) \quad (5.6)$$

respectively.

6. TENSOR SQUARES OF BOSE–MESNER ALGEBRAS

The tensor product of the left and right actions (5.5), (5.6) of A^0 on A^1 gives a ring homomorphism

$$L \otimes R: A^0 \otimes A^0 \rightarrow \text{End}_{\mathbb{C}} A^1. \quad (6.1)$$

The aim of this section is to prove that $L \otimes R$ embeds $A^0 \otimes A^0$ as a commutative subring of the endomorphism ring of the complex vector space A^1 . The proof depends on a lemma involving certain features of the superalgebra.

For a given function $f: \{1, \dots, m+2\} + \{1, \dots, n+2\}$, the contravariantly induced function $f^*: \Gamma^n \rightarrow \Gamma^m$ of Definition 4.2 (S2) extends to a unique linear function $f^*: A^n \rightarrow A^m$. In particular, $\lambda: \{1, 2, 3\} \rightarrow \{1, 2\}$ with $1\lambda = 1, 2\lambda = 3\lambda = 2$ and $\varrho: \{1, 2, 3\} \rightarrow \{1, 2\}$ with $1\varrho = 2\varrho = 1, 3\varrho = 2$ induce linear $\lambda^*: A^0 \rightarrow A^1$ and $\varrho^*: A^0 \rightarrow A^1$.

The elements of a given homogeneous component A^m are complex linear combinations $\sum c_i C_i^m$. Suppose that C_i^m is chosen to be $\{(x, x, \dots, x) \in Q^{m+2} | x \in Q\}$, i.e. $f^*(C_i^0)$ for any function $f: \{1, \dots, m+2\} \rightarrow \{1, 2\}$. Then the *trace function* $\text{tr}: A^m \rightarrow \mathbb{C}$ on A^m is defined by $\text{tr}(\sum c_i C_i^m) = |Q|^{m+1} c_1$. Note that for $m = 0$ this agrees with the usual trace function on the Bose–Mesner algebra A^0 , where the C_i^0 are identified with their incidence matrices.

LEMMA 6.1. *Let x, y be elements of A^0 .*

(i) *The equation*

$$\lambda^*(x) \cdot y = x \cdot \varrho^*(y) \quad (6.2)$$

holds in A^1 .

(ii) $\text{tr}(\lambda^*(x) \cdot y) = \text{tr}(x) \cdot \text{tr}(y) = \text{tr}(x \cdot \varrho^*(y)).$

PROOF. For (6.2), it suffices to prove

$$\lambda^*(C_i^0) \cdot C_j^0 = C_i^0 \cdot \varrho^*(C_j^0) \quad (6.4)$$

for given C_i^0, C_j^0 in Γ^0 . Fix C_k^1 in Γ^1 , with given element (x, y, z) . Then in the expansion of each side of (6.4) as a complex linear combination of elements of Γ^1 , the coefficient of C_k^1 is

$$|\{y \in Q | (x, y) \in C_i^0, (y, z) \in C_j^0\}|, \quad (6.5)$$

which is zero unless both $(x, y) \in C_i^0$ and $(y, z) \in C_j^0$, in which case it is 1. Setting $k = 1$ shows $\text{tr}(\lambda^*(C_i) \cdot C_j) = \delta_{ij}\delta_{ij}$, whence (6.3) follows by linearity.

THEOREM 6.2 *The algebra homomorphism $L \otimes R: A^0 \otimes A^0 \rightarrow \text{End}_{\mathbb{C}} A^1$ embeds the tensor square $A^0 \otimes A^0$ as a commutative subalgebra of the endomorphism ring $\text{End}_{\mathbb{C}} A^1$.*

PROOF. The commutative algebra A^0 has a basis $\{E_1^0, \dots, E_{s_0}^0\}$ of orthogonal idempotents with $E_i^0 = \sum \eta_{ij} C_j^0$ and $\text{tr}(E_i^0) = f_i \cdot |Q|^{-1} \neq 0$ (see [2, II.3.3], [6, §2]). Thus $A^0 \otimes A^0$ has $\{E_i^0 \otimes E_j^0 \mid 1 \leq i, j \leq s_0\}$ as a basis of orthogonal idempotents. Suppose that $\alpha = \sum c_{ij} E_i^0 \otimes E_j^0$ lies in the kernel of $L \otimes R$. Then for each element α of A^1 , $0 = \alpha L \otimes R(\alpha) = \sum c_{ij} E_i^0 \cdot \alpha \cdot E_j^0$. Given $1 \leq i, j \leq s$, use (6.2) to take $a_{ij} = \lambda^*(E_i^0) \cdot E_j^0 = E_i^0 \cdot \varrho^*(E_j^0)$. Then for $k \neq i$ or $l \neq j$, one has $E_k^0 \cdot a_{ij} \cdot E_l^0 = 0$. Thus $0 = a_{ij} L \otimes R(\alpha) = c_{ij} E_i^0 \cdot a_{ij} \cdot E_j^0 = c_{ij} E_i^0 E_i^0 \cdot \varrho^*(E_j^0) \cdot E_j^0 = c_{ij} a_{ij} \cdot E_j^0 = c_{ij} \lambda^*(E_i^0) \cdot E_j^0 E_j^0 = c_{ij} a_{ij}$. But $a_{ij} \neq 0$, since by (6.3) one has $\text{tr}(a_{ij}) = \text{tr}(E_j^0) \neq 0$. Thus $c_{ij} = 0$ for all $1 \leq i, j \leq s$. In other words, $\alpha = 0$ and $L \otimes R$ injects.

7. TENSOR SQUARES OF QUASIGROUP CHARACTER TABLES

Let Q be a finite non-empty quasigroup with character table Ψ . A generally valid combinatorial interpretation of the tensor square $\Psi \otimes \Psi$ of the character table may be given in terms of a superscheme associated with the quasigroup. The partition Γ^n of the superscheme is the set of orbits of the multiplication group G in its diagonal action $g: (x_1, \dots, x_{n+2}) \mapsto (x_1 g, \dots, x_{n+2} g)$ on Q^{n+2} . The associated scheme is the quasigroup conjugacy class partition of Q^2 [6, Theorem 3.1] having the centralizer ring $V(G, Q)$ as its Bose–Mesner algebra. By Theorem 6.2, the tensor square $V(G, Q) \otimes V(G, Q)$ embeds into the endomorphism ring of the homogeneous component A^1 of the Bose–Mesner superalgebra, according to the left and right actions of $V(G, Q) = A^0$ on A^1 described in Corollary 5.2. The bases

$$\{C_i^0 \otimes C_j^0 \mid 1 \leq i, j \leq s_0\} \quad \text{and} \quad \{E_l^0 \otimes E_m^0 \mid 1 \leq l, m \leq s_0\}$$

of $V(G, Q) \otimes V(G, Q)$ are related by

$$C_i^0 \otimes C_j^0 = \sum_{l=1}^{s_0} \sum_{m=1}^{s_0} \xi_{il} \xi_{jm} E_l^0 \otimes E_m^0. \quad (7.1)$$

The entries $\psi_{il} \psi_{jm}$ of $\Psi \otimes \Psi$ are then the normalized coefficients

$$\psi_{il} \psi_{jm} = (f_i f_j)^{1/2} \xi_{il} \xi_{jm} (n_i n_m)^{-1} \quad (7.2)$$

(cf. [6, Definition 3.3]). The combinatorial interpretation may thus be summarized as:

THEOREM 7.1. *Let Q be a finite non-empty quasigroup with multiplication group G and character table Ψ . Then the tensor square $\Psi \otimes \Psi$ is determined by the two-sided action of $V(G, Q)$ on the orbits of G on Q^3 .*

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